Regularity criteria in weak L^3 for 3D incompressible Navier-Stokes equations

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Abstract

We study the regularity of a distributional solution (u, p) of the 3D incompressible evolution Navier-Stokes equations. Let B_r denote concentric balls in $\mathbb{R}^3$ with radius r. We will show that if $p \in L^m(0,1; L^1(B_1))$, $m > 2$, and if u is sufficiently small in $L^\infty(0,1; L^{3,\infty}(B_1))$, without any assumption on its gradient, then u is bounded in $B_{1-\tau} \times (\tau, 1)$ for any $\tau > 0$. It is a borderline case of the usual Serrin-type regularity criteria, and extends the steady-state result of Kim-Kozono to the time dependent setting.
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This is a joint work with Yuwen Luo.

Outline

1. Background
2. Results
3. Sketch of proof

Part 1

Background

3D incompressible Navier-Stokes equations (NS)

Denote $B_r = \{x \in \mathbb{R}^3 : |x| < r\}$ and $D = B_1 \times (0,1)$

Consider (NS) for velocity and pressure $(u,p) : D \to \mathbb{R}^3 \times \mathbb{R}$,

$$\partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0, \quad \operatorname{div} u = 0, \tag{1}$$

in D, with $\quad \partial_t = \frac{\partial}{\partial t}, \quad u \cdot \nabla = \sum_{j=1}^3 u_j \partial_j, \quad \partial_j = \frac{\partial}{\partial x_j}$

Goal Interior regularity of a given distributional solution (u,p) in D satisfying

$$u \in L^\infty(0,1; L^{3,\infty}(B_1)), \quad \text{small}$$
$$p \in L^{2+\varepsilon}(0,1; L^1(B_1)), \quad \text{large ok}$$

Above $L^{3,\infty}$ = weak L^3, a member of Lorentz spaces $L^{q,r}$.
Recall $L^{3,1} \subset L^3 \subset L^{3,\infty}$ and $1/|x| \in L^{3,\infty}$.

Distributional solution versus very weak solution

$(u, p) \in L^2(D; \mathbb{R}^3) \times L^1(D)$ is a distributional solution of NS if

$$\iint_D \Big(u \cdot (-\partial_t \zeta - \Delta \zeta) - \sum u_i u_j \partial_i \zeta_j - p \operatorname{div} \zeta \Big) \, dxdt = 0 \qquad (2)$$

for any $\zeta \in C_c^2(D; \mathbb{R}^3)$, and

$$\int_{B_1} u(x,t) \cdot \nabla \phi(x) dx = 0, \quad \forall \phi \in C_c^1(B_1), \quad a.e. \ t. \qquad (3)$$

$u \in L^2(D; \mathbb{R}^3)$ is a very weak solution of NS if

$$\iint_D \Big(u \cdot (-\partial_t \zeta - \Delta \zeta) - \sum u_i u_j \partial_i \zeta_j \Big) \, dxdt = 0 \qquad (4)$$

for any $\zeta \in C_c^2(D; \mathbb{R}^3)$ with $\operatorname{div} \zeta = 0$, and (3) holds.

(continued)

- No initial or boundary condition. No ∇u assumption.
- No pressue for very weak solution.
- A distributional solution is automatically a very weak solution.
- The converse is false, e.g.

$$u = g(t)\nabla h(x), \quad p = -g'(t)h(x) - \frac{1}{2}g^2|\nabla h|^2 \tag{5}$$

with $\Delta h = 0$, $g \in L^\infty$, and $g' \notin L^1$.

A very weak solution u is called a weak solution if it is also in the energy class

$$u \in L^\infty(0,1;L^2) \cap L^2(0,1;H^1)$$

Regularity

For $z_0 = (x_0, t_0)$, denote $Q(z_0, r) = B_r(x_0) \times (t_0 - r^2, t_0)$.

We say u is regular at z_0 if $u \in L^\infty(Q(z_0, r))$ for some $r > 0$.

(It usually implies u is smooth in x.)

Open problem: Regularity of weak solutions.

Serrin-type regularity criteria for weak solutions

A weak solution u in $B_1 \times (0,1)$ is regular in $B_{1-\tau} \times (\tau, 1)$, $\forall \tau > 0$, if

$$u \in L^s L^q = L^s(0,1; L^q(B_1))$$

with

1. subcritical case: $\frac{3}{q} + \frac{2}{s} < 1$ (Serrin 62)
2. borderline cases: $\frac{3}{q} + \frac{2}{s} = 1$, $q > 3$ (Ladyzhenskaya, Giga, Sohr, Struwe)
3. endpoint case: $q = 3$, $s = \infty$ (Escauriaza-Seregin-Sverak 02)

(continued)

In Lorentz spaces, a weak solution u is regular if

1. $u \in L^s L^{q,\infty}$, $\frac{3}{q} + \frac{2}{s} = 1$, $q > 3$ (Takahashi 90)
2. $u \in L^{s,r} L^{q,\infty}$, $\frac{3}{q} + \frac{2}{s} = 1$, $q > 3$ (Sohr 01)
3. $|u(x,t)| \leq \frac{\varepsilon}{|x|^{1-\theta}|t|^{\theta/2}}$, $0 < \theta < 1$ (Chen-Price 01)
4. $u \in L^{s,r} L^{q,\infty}$ and small, $\frac{3}{q} + \frac{2}{s} = 1$, $q \geq 3$ (Kim-Kozono 04)

Enemy of regularity

Self-similar blow-up rate

$$|u(x,t)| \geq \frac{C_*}{|x| + \sqrt{T-t}}$$

$$\|\mathrm{RHS}\|_{L^s L^q} = \infty$$

$$\|\mathrm{RHS}\|_{L^{s,r} L^{q,\infty}} = CC_*$$

Smallness kills the enemy.

Regularity results for distributional solutions

For distributional solutions not in the energy class, the only known results are for stationary solutions:

1. Removable singularity (Dyer-Edmunds, Shapiro, Choe-Kim)
   If $(u, p)(x)$ distributional solution in $B_1 \backslash \{0\}$ with
   - $u(x) = o(|x|^{-1})$ as $|x| \to 0$, OR
   - $u \in L^3(B_1)$,

   then (u, p) solution in B_1 with $u \in L^\infty_{loc}$
2. Regularity (Kim-Kozono 06)
   If $(u, p) \in L^{3,\infty} \times L^1$ with $\|u\|_{L^{3,\infty}}$ small, then $u \in L^\infty_{loc}$.

(continued)

3. Landau solution (optimality) (Slezkin, Landau)

   For any $b \in \mathbb{R}^3$, there are U^b, P^b minus-one homogeneous solution in $\mathbb{R}^3$ of

   $$-\Delta u + u \cdot \nabla u + \nabla p = b\delta_0, \quad \operatorname{div} u = 0.$$

4. Asymptotics (Miura-Tsai 2012)

   For any $0 < \alpha < 1$, if $u(x)$ very weak solution in $B_1 \backslash \{0\}$ with $|u(x)| \leq \frac{\varepsilon}{|x|}$, $\varepsilon < \varepsilon_\alpha$, then for some U^b,

   $$|u(x) - U^b(x)| \leq \frac{C\varepsilon}{|x|^\alpha}$$

Time-dependent distributional solution

No previous results.

Motivation For $\Omega = \mathbb{R}^3$ or exterior domain, the class

$$u \in L^\infty(0, \infty; L^{3,\infty}(\Omega)) \tag{6}$$

contains stationary and time-periodic solutions, and is the natural space for self-similar solutions and for solutions with non-decaying boundary data. However,

1. in the class (6) one cannot construct weak solutions. One can construct mild solutions, which are not in the energy class even locally;
2. we need local pointwise bound to study, e.g., spatial decay of time-periodic solutions. (Kang-Miura-Tsai 2012)

Criticality of $L^\infty L^{3,\infty}$

NS in $\mathbb{R}^3$ with zero initial data is equivalent to $u = \Gamma(u)$,

$$\Gamma(u)_i(x,t) := -\int_0^t \int_{\mathbb{R}^3} (\partial_k S_{ij}(x-y,t-s))(u_j u_k)(y,s)dyds$$

where S_{ij} is the fundamental solution of the Stokes system in $\mathbb{R}^3$,

$$|D_x^\ell S(x,t)| \lesssim (|x| + \sqrt{t})^{-3-\ell}$$

By generalized Young inequality for convolution, if $\frac{3}{q} + \frac{2}{s} = 1$,

$$\|\Gamma(u)\|_{L^s L^q} \lesssim \|u\|_{L^s L^q}^2 .$$

It is not applicable at the endpoint $(q,s) = (3,\infty)$.

Part 2

Results

Main Theorem

Theorem 1.

There is $\varepsilon_1 > 0$ such that, if (u, p) distributional solution of NS in $B_1 \times (0,1)$ and for some $m > 2$,

$$u \in L^\infty(0,1; L^{3,\infty}(B_1)),$$
$$\|u\|_{L^\infty L^{3,\infty}} \le \varepsilon_1,$$
$$p \in L^m(0,1; L^1(B_1)), \quad \text{large ok}$$

then $u \in L^\infty(B_{1-\tau} \times (\tau, 1))$ for any $\tau > 0$.

Remarks:

(i) End point case, smallness on u, not on p.
ε_1 independent of m and $\|p\|$.

(ii) Compare Kim-Kozono 2004 for weak solution in energy class
(no assumption on p, but $\nabla u \in L^2 L^2$)

(iii) It extends Kim-Kozono 2006 stationary case.

(iv) The integrability of p is mild, but nontrivial unless $\Omega = \mathbb{R}^3$.

Lemma. (subcritical case)
Suppose for $3 < q \le \infty$, $3 \le s \le \infty$, $\frac{3}{q} + \frac{2}{s} < 1$,

$$u \in L^s(0,1; L^q(B_1)) \cap L^\infty(0,1; L^1(B_1))$$

is a very weak solution, then $u \in L^\infty(B_{1-\tau} \times (\tau, 1))$ for any $\tau > 0$.

Theorem 1'.
Suppose for $3 < q \le \infty$, $3 \le s \le \infty$, $\frac{3}{q} + \frac{2}{s} \le 1$, $m \ge 1$,
$m > \frac{2q}{3(q-3)}$,

$$u \in L^s(0,1; L^q(B_1)) \cap L^\infty(0,1; L^1(B_1)), \quad p \in L^m(0,1; L^1(B_1))$$

is a distributional solution of NS in $B_1 \times (0,1)$, then
$u \in L^\infty(B_{1-\tau} \times (\tau, 1))$ for any $\tau > 0$.

Stationary case

Theorem 2. [Kim-Kozono]
There is $\varepsilon_2 > 0$ such that, if (u, p) distributional solution of NS in Ω with $p \in L^1$ and $\|u\|_{L^{3,\infty}} \leq \varepsilon_2$, then $u \in L^\infty_{loc}$.

Theorem 3.
There is $\varepsilon_3 > 0$ such that, if u very weak solution of NS in Ω with $\|u\|_{L^{3,\infty}} \leq \varepsilon_3$, then $u \in L^\infty_{loc}$ and

$$|u(x)| \leq \frac{C}{\mathrm{dist}(x, \partial\Omega)} \|u\|_{L^{3,\infty}}$$

Part 3

Sketch of proof

1. Linear estimate

Yamazaki: For $1 < p \le q < \infty$, ($q < 3$ for exterior domain)

$$\int_0^\infty t^{\frac{3}{2p}-\frac{3}{2q}-\frac{1}{2}} \left\| \nabla e^{t\Delta}\zeta \right\|_{L^{q,1}(\mathbb{R}^3)} dt \le C \left\|\zeta\right\|_{L^{p,1}_\sigma(\mathbb{R}^3)}$$

Thus the solution operator for Stokes system in $\mathbb{R}^3 \times (0,\infty)$

$$\partial_t v - \Delta v + \nabla p = \operatorname{div} F, \quad \operatorname{div} v = 0, \quad v|_{t=0} = 0$$

defined by duality for $\zeta \in C_c^\infty$, $\operatorname{div}\zeta = 0$,

$$((\Phi F)(t),\zeta) = \int_0^t (-F_{jk}(t-\tau), \partial_j(e^{\tau\Delta}\zeta)_k)d\tau,$$

maps $F \in L^\infty(0,\infty; L^{s,\infty})$ to $\Phi F \in C_w([0,\infty), L_\sigma^{s^*,\infty})$, $1 < s < 3$.

Remark. For Theorem 1', Φ is defined by usual convolution and pointwise estimates for fundamental solution.

2. Reformulation

Extension with cut-off $\varphi(x,t)$

$$\tilde{u} = u\varphi + \nabla\eta, \quad \tilde{p} = \varphi p - \partial_t\eta + \Delta\eta, \quad \eta = \frac{1}{4\pi|x|} * (\nabla\varphi \cdot u),$$

satisfies in $\mathbb{R}^3 \times (0,1)$

$$\partial_t v - \Delta v + \nabla q = f^0 + \operatorname{div}(F^1 - \tilde{\varphi} u \otimes v), \quad \operatorname{div} v = 0, \quad v|_{t=0} = 0,$$

which is equivalent to the linear operator equation in v

$$v = v^0 - \Phi(\tilde{\varphi} u \otimes v) \tag{7}$$

where v^0 defined by convolution of fundamental solution with the source term $f^0 + \operatorname{div} F^1$.

3. Existence of regular solution

- For some $\delta = \delta(m) > 0$,

$$v_0 \in C_w([0,1]; L^{3,\infty} \cap L^{3+\delta,\infty}(\mathbb{R}^3))$$

- Equation (7) has a mild solution

$$v \in C_w([0,1]; L^{3,\infty} \cap L^{3+\delta,\infty}(\mathbb{R}^3))$$

It is also a very weak solution and, by Lemma, bounded for $t > 0$.

4. Uniqueness of less regular solution

- Our $\tilde{u} = u\varphi + \nabla\eta$

$$\tilde{u} \in L^\infty(0,1; L^{3,\infty}(\mathbb{R}^3))$$

- Solution v of (7) is unique in the class

$$v \in L^\infty(0,1; L^{3,\infty}(\mathbb{R}^3))$$

- Thus the regular solution in Step 3 agrees with $\tilde{u}$, which equals u in $B_{1-\tau} \times (\tau, 1)$.